

# A CHARACTERIZATION OF THE BALL

K. DIEDERICH, J. E. FORNÆSS, AND E. F. WOLD

ABSTRACT. We study bounded domains with certain smoothness conditions and the properties of their squeezing functions in order to prove that the domains are biholomorphic to the ball.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ . For  $z \in \Omega$  let  $f_z : \Omega \rightarrow \mathbb{B}(0, 1)$  be any 1-1 holomorphic map to the unit ball which maps  $z$  to the origin. Let  $S_{\Omega, f_z}(z) = \sup\{r > 0; \mathbb{B}(0, r) \subset f_z(\Omega)\}$ . We define the squeezing function  $S = S_\Omega : \Omega \rightarrow (0, 1]$  by setting  $S(z) = \sup_{f_z} \{S_{\Omega, f_z}\}$ . See [DGZ1], [DGZ2], [KZ], [LSY1], [LSY2], [Ye] and references therein for results on the squeezing function.

In [FW] Fornæss and Wold proved the following estimate for strongly pseudoconvex domains with smooth boundary.

**Theorem 1.1.** *Let  $\Omega$  be a bounded strongly pseudoconvex domain with  $\mathcal{C}^4$  boundary in  $\mathbb{C}^n$ . Then there exists a constant  $C > 0$  so that the squeezing function  $S_\Omega(z)$  satisfies the estimate  $S_\Omega(z) \geq 1 - Cd(z)$  on  $\Omega$  where  $d(z)$  denotes the boundary distance.*

Here we show that this estimate is sharp: Recall that the squeezing function of the unit ball is identically equal to 1. In fact if the squeezing function has the value one at at least one point, then the domain is known to be biholomorphic to the ball.

**Theorem 1.2.** *Let  $\Omega$  be a bounded domain with  $\mathcal{C}^2$  boundary in  $\mathbb{C}^n$ . Suppose there does not exist a constant  $c > 0$  so that the squeezing function  $S_\Omega(z)$  satisfies the estimate  $S_\Omega(z) \leq 1 - cd(z)$  on  $\Omega$ . Then  $\Omega$  is biholomorphic to the ball.*

In the second section, we prove Theorem 1.2. In the third section we show that the theorem fails for domains with only  $\mathcal{C}^1$  boundary.

---

<sup>1</sup>The second author was supported in part by the Norwegian Research Council grant number 240569 and NSF grant DMS1006294

<sup>2</sup>The third author was supported in part by the Norwegian Research Council grant number 240569

Date: April 19, 2016.

2010 *Mathematics Subject Classification.* 32A99.

*Key words and phrases.* Squeezing function and its boundary behaviour, Implications for characterizing the ball.

## 2. PROOF OF THE THEOREM

Theorem 1.3 is equivalent to the following result:

**Theorem 2.1.** *Let  $\Omega$  be a bounded domain with  $\mathcal{C}^2$  boundary. Suppose there is a sequence of points  $p_i$  approaching the boundary so that the squeezing function  $S(p_i) \geq 1 - \epsilon_i d(p_i)$ ,  $\epsilon_i \rightarrow 0$ . Then  $\Omega$  is biholomorphic to the ball.*

*Proof.* Say  $0 \in \Omega$ . Let  $\Phi_i : \Omega \rightarrow \mathbb{B}(0, 1)$  be 1-1 holomorphic maps so that  $\Phi_i(p_i) = 0$  and the image contains the ball of radius  $S(p_i)$ .

We collect some lemmas.

**Lemma 2.2.** *If  $\Omega$  is a bounded domain with  $\mathcal{C}^2$  boundary, then there is a constant  $C$  so that the Kobayashi distance from 0 to  $p_i$  satisfies the estimate  $d_{K,\Omega}(0, p) \leq \frac{1}{2} \log \frac{1}{d(p)} + C$*

We prove this by choosing a curve from 0 to  $p_i$  which ends as a straight normal line at  $p_i$ . Then we compare the infinitesimal Kobayashi metric on  $\Omega$  with the metric on the intersection with the complex normal line.

Recall the Kobayashi distance on the unit ball:

$$d_{K,\mathbb{B}(0,1)}(0, z) = \frac{1}{2} \log \frac{1 + \|z\|}{1 - \|z\|}.$$

**Lemma 2.3.** *For points  $z \in B(0, 1 - \epsilon_i d(p_i))$ , we have that*

$$\frac{1}{2} \log \frac{1 + \|z\|}{1 - \|z\|} \leq d_{K,\Phi_i(\Omega)}(0, z)$$

**Lemma 2.4.** *For all  $i$ ,  $\Phi_i(0) \in \mathbb{B}(0, 1 - d(p_i)/e^{2C})$ .*

*Proof.* Let  $\|z\| = 1 - d(p_i)/e^{2C}$ .

Then

$$\begin{aligned}
d_{K, \Phi_i(\Omega)}(0, z) &\geq \frac{1}{2} \log \frac{1 + \|z\|}{1 - \|z\|} \\
&= \frac{1}{2} \log \frac{1 + (1 - \frac{d(p_i)}{e^{2C}})}{1 - (1 - \frac{d(p_i)}{e^{2C}})} \\
&= \frac{1}{2} \log \frac{2 - \frac{d(p_i)}{e^{2C}}}{\frac{d(p_i)}{e^{2C}}} \\
&> \frac{1}{2} \log \frac{e^{2C}}{d(p_i)} \\
&= \frac{1}{2} \log \frac{1}{d(p_i)} + C \\
&\geq d_{K, \Omega}(0, p_i) \\
&= d_{K, \Phi_i(\Omega)}(0, \Phi_i(0))
\end{aligned}$$

Hence any path connecting 0 to  $\Phi_i(0)$  which passes through a point on the boundary of the ball  $\mathbb{B}(0, 1 - d(p_i)/e^{2C})$  is too long compared to the Kobayashi distance from 0 to  $\Phi_i(0)$ .

□

We can assume that  $\Phi_i(0) = (r, 0, \dots, 0)$ ,  $0 \leq r < 1 - d(p_i)/C$ . Define

$$\Psi_i(z_1, \dots, z_n) = \left( \frac{z_1 - r}{1 - z_1 r}, \frac{\sqrt{1 - r^2} z_2}{1 - z_1 r}, \dots, \frac{\sqrt{1 - r^2} z_n}{1 - z_1 r} \right).$$

Then  $\Psi_i$  is an automorphism of the unit ball and the map  $F_i = \Psi_i \circ \Phi_i$  is a 1-1 holomorphic map on  $\Omega$  into the unit ball which maps 0 to 0.

**Lemma 2.5.**  $F_i(\Omega) \supset \mathbb{B}(0, 1 - 6C\epsilon_i)$ .

*Proof.* We know that  $\Phi_i(\Omega) \supset \mathbb{B}(0, 1 - \epsilon_i d(p_i))$ . To prove the lemma it suffices to prove that if  $\|z\| = 1 - 2\epsilon_i d(p_i)$ , then  $\|\Psi_i(z)\| \geq 1 - 6C\epsilon_i$ . Suppose that  $\|z\| = 1 - 2\epsilon_i d(p_i)$ . Then

$$\begin{aligned}
\|\Psi_i(z)\|^2 &= \frac{(z_1 - r)(\bar{z}_1 - r) + (1 - r^2)(|z_2|^2 + \cdots + |z_n|^2)}{|1 - z_1 r|^2} \\
&= \frac{(z_1 - r)(\bar{z}_1 - r) + (1 - r^2)(1 - 2\epsilon_i d(p_i)^2 - |z_1|^2)}{|1 - z_1 r|^2} \\
&= \frac{(z_1 - r)(\bar{z}_1 - r) + (1 - r^2)(1 - |z_1|^2)}{|1 - z_1 r|^2} \\
&+ \frac{(1 - r^2)(-4\epsilon_i d(p_i) - 4\epsilon_i^2 d^2(p_i))}{|1 - z_1 r|^2} \\
&= 1 - \frac{(1 - r^2)(4\epsilon_i d(p_i) + 4\epsilon_i^2 d^2(p_i))}{|1 - z_1 r|^2} \\
&\geq 1 - \frac{(1 - r^2)(5\epsilon_i d(p_i))}{(1 - r)^2} \\
&\geq 1 - \frac{10\epsilon_i d(p_i)}{1 - r} \\
&\geq 1 - \frac{10C\epsilon_i d(p_i)}{d(p_i)} \\
&= 1 - 10C\epsilon_i \\
&\Rightarrow \\
\|\Psi_i(z)\| &\geq 1 - 6C\epsilon_i
\end{aligned}$$

□

**Corollary 2.6.**  $S(0) = 1$

**Corollary 2.7.**  $\Omega$  is biholomorphic to the unit ball.

□

### 3. AN EXAMPLE

Let  $\Omega'$  be a  $\mathcal{C}^\infty$  domain in the right half plane where the boundary contains an interval  $(-i, i)$  on the imaginary axis and which is a topological annulus. We define  $\Omega = \Phi(\Omega')$  there  $\Phi(z) = z \log z$ . The squeezing function on  $\Omega'$  satisfies the estimate  $S(z) \geq 1 - Cd(z)$  since  $\Omega'$  is strongly pseudoconvex. The squeezing function is a biholomorphic invariant and the derivative of  $\Phi$  goes to zero when we approach the origin. Hence the squeezing function of  $\Omega$  will not satisfy the estimate  $S_\Omega \leq 1 - cd$  for any  $c > 0$ . However, the domain is a topological annulus so cannot be biholomorphic to the ball. This shows that Theorem 1.2 fails if we only assume that the boundary is  $\mathcal{C}^1$ .

## REFERENCES

- [DGZ1] F. Deng, Q. Guan and L. Zhang, *Some properties of squeezing functions on bounded domains*, Pacific J. Math. **257** (2012), 319 - 341.
- [DGZ2] F. Deng, Q. Guan and L. Zhang, *Properties of squeezing functions and global transformations of bounded domains*, Trans. Amer. Math. Soc. **368** (2016), 2679 - 2696.
- [FW] J. E. Fornæss., E. F. Wold.; *An estimate for the squeezing function and estimates of invariant metrics*, arXiv:1411.3846 (2014)
- [KZ] K.-T. Kim and L. Zhang, *On the uniform squeezing property of convex domains in  $\mathbb{C}^n$* , Pacif. J. Math., **282** (2016), 341 - 358.
- [LSY1] K. Liu, X. Sun, and S.-T. Yau, *Canonical metrics on the moduli space of Riemann surfaces, I*, J. Differential Geom. **68** (2004), 571 - 637.
- [LSY2] K. Liu, X. Sun, and S.-T. Yau, *Canonical metrics on the moduli space of Riemann surfaces, II*, J. Differential Geom. **69** (2005), 163 - 216.
- [Ye] S.-K. Yeung, *Geometry of domains with the uniform squeezing property*, Adv. Math. **221** (2009), 547 - 569.

K. DIEDERICH: UNIVERSITET WUPPERTAL, MATHEMATIK, GAUSSTRASSE 20, 42119 WUPPERTAL, GERMANY.

*E-mail address:* `klas@math.uni-wuppertal.de`

J. E. FORNÆSS: DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY 7491 TRONDHEIM, NORWAY

*E-mail address:* `john.fornass@math.ntnu.no`

E. F. WOLD: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, POSTBOKS 1053 BLINDERN, 0316 OSLO, NORWAY

*E-mail address:* `erlendfw@math.uio.no`